Friday, October 2, 2015

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Problem 59

Problem. Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y = 1/x$, $y = 0$, $x = \frac{1}{4}$, and $x = c$ (where $c > \frac{1}{4}$) is revolved about the x-axis and the y-axis, respectively. Find the value of c for which $V_1 = V_2$.

Solution.

$$\begin{aligned} V_1 &= \int_{1/4}^{c} 2\pi \cdot \frac{1}{x^2}\, dx \\ &= -2\pi \left[\frac{1}{x}\right]_{1/4}^{c} \\ &= -2\pi \left(\frac{1}{c} - 4\right) \\ &= 2\pi \left(4 - \frac{1}{c}\right). \end{aligned}$$

$$\begin{aligned} V_2 &= \int_{1/4}^{c} 2\pi x \left(\frac{1}{x}\right)\, dx \\ &= 2\pi \int_{1/4}^{c} dx \\ &= 2\pi \left[x\right]_{1/4}^{c} \\ &= 2\pi \left(c - \frac{1}{4}\right). \end{aligned}$$

Now solve the equation $V_1 = V_2$ for c.

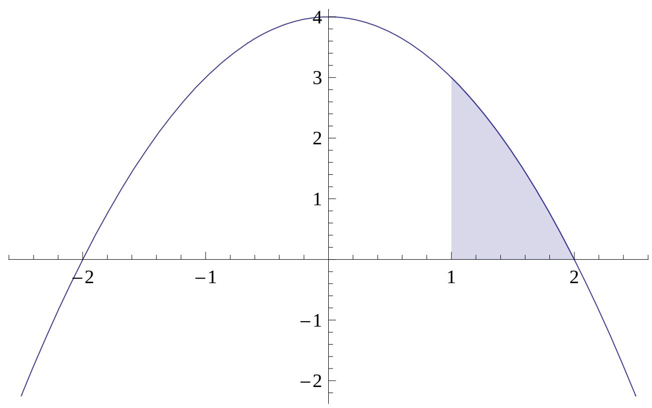
$$\begin{aligned} 2\pi \left(4 - \frac{1}{c}\right) &= 2\pi \left(c - \frac{1}{4}\right) \\ 4 - \frac{1}{c} &= c - \frac{1}{4} \\ 16c - 4 &= 4c^2 - c \\ 4c^2 - 17c + 4 &= 0 \\ (4c - 1)(c - 4) &= 0. \end{aligned}$$

The solutions are $c = \frac{1}{4}$ and $c = 4$. We were told that $c > \frac{1}{4}$, so the solution is $c = 4$.

Problem 60

Problem. The region bounded by $y = r^2 - x^2$, $y = 0$, and $x = 0$ is revolved about the y-axis to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0 < k < r$. Find the volume of the resulting ring (a) by integrating with respect to x and (b) by integrating with respect to y.

Solution. (a) The graph (with, e.g., $r = 2$ and $k = 1$) is



The figure is rotated about the y-axis, so if we integrate with respect to the x-axis, then we must use the shell method. The radius is x and the height is $r^2 - x^2$.

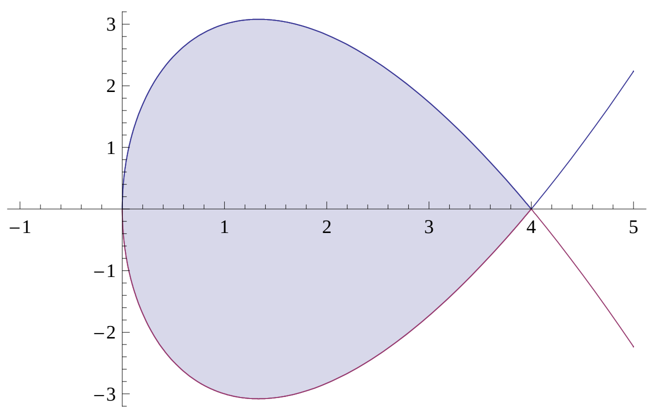
$$\begin{aligned} V &= \int_k^r 2\pi x(r^2 - x^2)\,dx \\ &= 2\pi \int_k^r (r^2 x - x^3)\,dx \\ &= 2\pi \left[\frac{1}{2}r^2x^2 - \frac{1}{4}x^4\right]_k^r \\ &= 2\pi \left(\left(\frac{1}{2}r^4 - \frac{1}{4}r^4\right) - \left(\frac{1}{2}r^2k^2 - \frac{1}{4}k^4\right)\right) \\ &= \frac{\pi}{2}\left(r^4 - 2r^2k^2 + k^4\right) \\ &= \frac{\pi}{2}(r^2 - k^2)^2. \end{aligned}$$

(b) To integrate with respect to y, we must use the washer method. The inner radius is $R_1 = k$ and the outer radius is $R_2 = \sqrt{r^2 - y}$. The limits of integration are from 0 to $r^2 - k^2$.

$$\begin{aligned} V &= \int_0^{r^2-k^2} \pi\left((r^2 - y) - k^2\right)\, dy \\ &= \pi\left[(r^2 - k^2)y - \frac{1}{2}y^2\right]_0^{r^2-k^2} \\ &= \pi\left((r^2 - k^2)^2 - \frac{1}{2}(r^2 - k^2)^2\right) \\ &= \frac{\pi}{2}(r^2 - k^2)^2. \end{aligned}$$

Problem 61

Problem. Consider the graph of $y^2 = x(4 - x)^2$. Find the volumes of the solids that are generated when the loop of this graph is revolved about (a) the x-axis, (b) the y-axis, and (c) the line $x = 4$.



Solution. Solutions coming soon...

The answers are

(a) $\frac{512\pi}{3}$

(b) $\frac{2048\pi}{35}$

(c) $\frac{8192\pi}{105}$